Power-law tail probabilities of drainage areas in river basins

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We examine the appearance of power-law behavior in rooted tree graphs in the context of river networks. It has long been observed that the tails of statistical distributions of upstream areas in river networks, measured above every link, obey a power-law relationship over a range of scales. We examine this behavior by considering a subset of all links, defined as those links which drain complete Strahler basins, where the Strahler order defines a discrete measure of scale, for self-similar networks with both deterministic and random topologies. We find an excellent power-law structure in the tail probabilities for complete Strahler basin areas, over many ranges of scale. We show analytically that the tail probabilities converge to a power law under the assumptions of (1) simple scaling of the distributions of complete Strahler basin areas and (2) application of Horton's law of stream numbers. The convergence to a power law does not occur for all underlying distributions, but for a large class of statistical distributions which have specific limiting properties. For example, underlying distributions which are exponential and gamma distributed, while not power-law scaling, produce power laws in the tail probabilities when rescaled and sampled according to Horton's law of stream numbers. The power-law exponent is given by the expression $\phi=\ln(R_b)/\ln(R_A)$, where R_b is the bifurcation ratio and R_A is the Horton area ratio. It is commonly observed that $R_b \approx R_A$ in many river basins, implying that the tail probability exponent for complete Strahler basins is close to 1.0.

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I. INTRODUCTION

Scaling theories play an important role in quantification of scale invariance in physical systems. From the classical studies of turbulence [1,2] to complex system analyses [3,4], the appearance of power-law behavior has been seen as the signature of scale invariance. Theoretical investigations of the scaling properties of natural hydrological and geomorphological systems have provided a wealth of insights into the development of complex system analyses and the description of emergent patterns. Geomorphology provides the basis for this paper, but the results presented here may be applicable to a large variety of other physical systems.

River network topology can be modeled as rooted binary tree graphs, where water flows from small branches into larger branches across a large range of spatial scales. Networks are composed of links and nodes: Links in the tree graph represent individual river channels and, when two channels combine at their downstream ends, they form a node. Nodes have two upstream links draining directly into them and one link draining out. The entire collection of links and nodes forms a treelike structure, with no loops, so that there is a unique path between any two links in the network. Strahler streams, or simply streams, are composed of collections of continuous links and are defined according to the Horton-Strahler ordering convention. This convention assigns an *order* to each stream, which we use as a representation of the scale of the stream. The leaves or sources (links which do not have any links upstream of them) are defined to be of order 1. Higher-order streams are defined in terms of the orders of the streams which drain into them. When two streams, which have same orders, combine at a node, the link downstream of that node is one order higher. When two streams of different orders combine, the downstream link takes the greater order of the upstream links. Hence, if two order-1 streams come together, they form an order-2 stream. If an order-2 stream and an order-1 stream come together, the downstream link is order 2, extending the stream. Complete Strahler basins are sub-basins which drain directly into a basin that has the higher order.

The scaling properties of topologic and geometric network properties have been extensively studied in river basins. Models of river networks have been developed based on diffusion limited aggregation [5–7], random walk models [8–10], thermodynamics through principles of optimization [11–14], as well as other physically based models. In addition, power-law scaling exponents have been derived in the context of fractal geometry of river networks [15,16]. Various measures of scale invariance have been utilized in the

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analysis of drainage patterns, particularly in the context of self-similarity and self-affinity. Some measures of scale invariance deal with the scaling behavior of network branching structure, or network topology, independent of geometric constraints. Scale invariance in network topology is referred to as topological self-similarity, and has been formulated in terms of both the average branching structure (mean self-similarity or Tokunaga self-similarity) and the scaling of distributions of the branching structure (statistical self-similarity).

There are three major analytic topological theories of river networks which exhibit topological self-similarity, namely, the well-known random topology model [17,18], the Tokunaga self-similar network model [19,20], and the random self-similar network (RSN) model [21,22]. The random topology model is a special case of the more general Galton-Watson branching processes [23], while the Tokunaga model is based on the average self-similar behavior of river networks, and is a special case of the more general class of mean self-similar networks [21].

The RSN model generates topologically variable networks that exhibit mean topological self-similarity and, as a special case, the Tokunaga self-similarity. In addition, networks generated by the RSN model show simple scaling of the distributions of many topologic and geometric variables, with the Strahler order as the appropriate scale parameter. The Strahler ordering is a natural way to collect the links in a nested network into streams which represent different discrete scales. For instance, the RSN model predicts that the statistical distributions of the basin area at different scales (Strahler orders) have the same functional form, but are rescaled by a constant [22]. This type of scale invariance of statistical distributions is called distributional simple scaling (DSS). The existence of DSS for complete Strahler basin areas and the assumption that Horton's law of stream numbers (defined below) holds predicts that the tail probabilities of cumulative upstream areas of complete Strahler basins exhibit a power-law form with an exponent of $\ln(R_b)/\ln(R_A)$, and the demonstration of this result is the focus of this paper.

It has long been recognized that the ratio of the number of streams of successive orders converges to a constant. This is generally known as Horton's law of stream numbers. The law of stream numbers is an asymptotic result, with convergence in the following manner. Denote the number of streams of order ω in a basin of total order Ω as $N_\omega^{(\Omega)}$. Horton's law of stream numbers is then written as

$$\frac{N_\omega^{(\Omega)}}{N_{\omega+1}^{(\Omega)}} \to R_b, \quad \Omega \to \infty, \tag{1}$$

where R_b is the bifurcation ratio, which ranges from about 4.1 to 4.7 in natural river networks [24]. Equation (1) is valid for orders $\omega \ll \Omega$ in the limit the total basin order $\Omega \to \infty$. Note that Eq. (1) defines the Horton law of stream numbers as an asymptotic result, not in the limit $\omega \to \infty$; but rather there is a convergence of the ratio of the number of sub-basins which may be of low order, but are nested in an overall basin which gets very large. In practice, this limit is reached rapidly for basins with $\Omega > 5$ or so. We use Horton's law of stream numbers as expressed in Eq. (1) to sample rescaled distributions of basin area in our derivation of power-law tail probabilities in this paper. Horton's law of stream numbers is an example of the classic Horton laws, which relate the average values of various topologic, geometric, and geomorphologic variables at different scales. For instance, the Horton law for basin area is written in a form similar to Eq. (1),

$$\frac{\overline{A}_\omega^{(\Omega)}}{\overline{A}_{\omega-1}^{(\Omega)}} \to R_A, \quad \Omega \to \infty, \tag{2}$$

where $\overline{A}_\omega^{(\Omega)}$ is the average basin area at scale ω in a basin with total order Ω and R_A is the Horton area ratio.

Distributional simple scaling (DSS) implies that the statistical distribution of a given variable measured at a given scale is a rescaled version of some underlying distribution which is independent of scale. In the context of river basins, the Strahler order is a natural choice for scale parameter, and the rescaling constant has been shown to be the appropriate Horton ratio [22]. For instance, consider basin magnitude as a typical topological variable. The magnitude is the total number of first-order streams in the basin and can be considered as the topological equivalent of basin area. In a large river basin of total order Ω, there will be many sub-basins of lower order. Fixing a given order $\omega < \Omega$, the distribution of the basin magnitude $M_\omega^{(\Omega)}$ measured over all order ω sub-basins is a rescaled version of some underlying distribution, $\tilde{M}$, if the distributional simple scaling holds. Statistical distributions at different scales ω and k can then be related, since both are rescaled versions of $\tilde{M}$. We write

$$M_\omega^{(\Omega)} \overset{d}{=} R_M^{\omega-k} M_k^{(\Omega)}, \quad \omega,k < \Omega, \ \Omega \to \infty \tag{3}$$

to denote distributional simple scaling, where $\overset{d}{=}$ means equality in distribution and R_M is a rescaling constant independent of order ω [22]. There is good evidence that natural river networks exhibit distributional simple scaling for many topologic and geometric variables [25]. Classic Horton laws are derived by taking the expectation of equations, such as Eq. (3), and refers to relationships of means across scales rather than of entire distributions. Scale invariance of statistical distributions extends to the geometric and geomorphologic properties of river basins as well. For example, stream lengths and basin areas exhibit distributional simple scaling [25], as does the maximum of the width function [26] with the Strahler order playing the role of the scale parameter. DSS of hydrologic variables is a generalization of Horton's laws, which may be obtained by taking the expectation of equations such as Eq. (3).

It has also been observed that the probability of exceedence for cumulative upstream area above all links in a network, sometimes also referred to as the cumulative area distribution (CAD), exhibits power-law behavior, written as

$$P(A > a) \sim a^{-\theta}, \tag{4}$$

where A represents the area above a link chosen at random from all links in the network and θ is empirically observed to be close to 0.43 [27]. Power-law scaling of tail probabilities is generally seen as a signature of the scale invariance of topography in river basins [25,28,29]. The power-law behavior of tail probabilities for the area above all links will not be addressed in this paper, rather we focus our examination on the power-law behavior of exceedence probabilities for complete Strahler basins only. Complete Strahler basins are a subset of all sub-basins, consisting of basins which drain directly into streams that are of a higher order, and compose more than half of all sub-basins in a given basin. See Sec. II below.

We show below that the power-law behavior in the tail probabilities for complete Strahler basin areas (defined below) is also observed, written in a form similar to Eq. (4),

$$P(A_{\omega}^{(\Omega)}>a)\sim a^{-\phi}, \tag{5}$$

where $A_{\omega}^{(\Omega)}$ represents the area of a randomly chosen complete Strahler basin of order ω in a network of total order Ω, and the scaling exponent $\phi=\ln(R_b)/\ln(R_A)$. Model predictions of the random topology model, Tokunaga self-similar networks, and random self-similar networks predict that $R_b=R_A$, which implies that $\phi=1.0$. We show here that the power-law behavior in Eq. (5) is a consequence of Horton's law of stream numbers and distributional simple scaling of complete Strahler basin areas.

Recently, the authors of Refs. [30–32] examined the distributional scaling relationships in river networks. Among other results, they considered fluctuations and joint properties of basin areas and stream lengths, distributional properties of side-tributary statistics, and theoretical models of Hack's law and their relationship to the CAD power-law scaling. They also provide an analytic derivation of power-law scaling for main stream lengths under the assumption of exponential distributions of stream segment lengths. In contrast, we concentrate in this paper on the derivation of power-law scaling due to the distributional simple scaling of basin areas and Horton's law of stream numbers, independent of a particular choice of statistical distribution.

The authors of Ref. [33] observed tail probability scaling exponents near 1.0 where they only considered a subset of all basins. They defined main basins by considering links moving upstream from the outlet in the direction of the greatest contributing area. This selection method chooses essentially complete basins, but also includes links along the main streams of those basins. The existence of power-law scaling with an exponent near 1.0 for this subset of basins is possibly a further indication that the power-law scaling seen for all the links in a network is inherited from the more fundamental power-law scaling of complete Strahler basins derived here.

II. TAIL PROBABILITIES OF COMPLETE BASIN AREAS FOR DETERMINISTIC SELF-SIMILAR NETWORKS

We begin by examining the behavior of tail probabilities for complete Strahler basins in the case of deterministically generated self-similar networks. Deterministic self-similar networks have the property that the branching structure of all basins of a given order is identical. Thus, there is no way to discuss distributions as a function of scale for deterministic networks. However, we can use the identical nature of the topology to derive the scaling exponent for the tail probability. In the derivation below, we use a basin magnitude as a proxy for basin area.

Complete Strahler basins are defined as the basins upstream from the links which drain directly into the links that have a higher Strahler order. For instance, a side tributary to the main stream in a basin is a complete Strahler basin, because the Strahler order of the link at the outlet of the side tributary is less than the order of the main stream. However, sub-basins above a link in the main stream are not complete Strahler basins, because they drain into links which have the same Strahler order. This choice of restriction is not arbitrary. Complete basins represent naturally defined elementary units at a given discrete scale, defined by the Strahler order. This is important because, with the definition of a discrete scale parameter, it is possible to speak of statistical distributions of variables as a function of scale, as will be done in Sec. III. This is not possible if a continuous scale parameter, such as the basin magnitude, is used. It has been shown that the statistical distributions of the complete Strahler basin area exhibit distributional simple scaling [26].

The authors of Refs. [34,24] examined the topological equivalent of the tail probability exponent θ for basins upstream of every link in a network (the CAD exponent) for deterministic self-similar river networks. They found that the scaling exponent is given in terms of the Horton ratios for a number of links per stream, R_C, and the bifurcation ratio R_b as

$$\theta=1-\frac{\ln(R_C)}{\ln(R_b)}. \tag{6}$$

The argument in Ref. [24] uses the equivalence of probabilistic statements regarding the basin magnitude and the Strahler order in deterministic self-similar networks to derive Eq. (6). We extend the argument to complete Strahler basins below to show that for deterministic self-similar networks the complete basin tail probability scaling exponent is $\phi=1.0$.

Consider the probability that a complete Strahler basin, chosen at random in a deterministic self-similar network of total order Ω, has a basin magnitude M less than some threshold value m, where we have dropped the superscript (Ω) for notational simplicity. The probability is equal to the probability that the basin chosen has an order W less than the corresponding threshold order ω due to the deterministic nature of the network (the topology of every order ω network is identical). In other words, the two statements of probability

$$P(M<m)\Leftrightarrow P(W<\omega) \tag{7}$$

are equivalent, where M is the magnitude of a randomly chosen complete Strahler basin, W is the order of a randomly chosen complete basin, and m and ω are the thresholds for magnitude and order, respectively, against which the tail probabilities are calculated. The probability that a randomly chosen basin has an order less than some threshold ω is simply the total number of basins with order less than ω normalized by the total number of basins,

$$P(W<\omega)=\sum_{k=1}^{\omega-1} N_k^{(\Omega)} \Big/ \sum_{k=1}^{\Omega} N_k^{(\Omega)}. \tag{8}$$

Using the asymptotic expression for a number of streams,

$$N_k^{(\Omega)} \sim R_B^{\Omega-\omega+1},$$

valid for a large total network order Ω, Eq. (8) can be rewritten as

$$P(W<\omega)=\left[R_B^{\Omega+1}\sum_{k=1}^{\omega-1}\left(\frac{1}{R_B}\right)^k\right] \Big/ \left[R_B^{\Omega+1}\sum_{k=1}^{\Omega}\left(\frac{1}{R_B}\right)^k\right]. \tag{9}$$

Defining $p \equiv (1/R_B)$ Eq. (9) can be written in the standard form of a geometric series and summed to become

$$P(W<\omega)=\left[\sum_{k=0}^{\omega-1} p^k-1\right] \Big/ \left[\sum_{k=0}^{\Omega} p^k-1\right] \tag{10}$$

$$=\left[\frac{1-p^{\omega}}{1-p}-1\right] \Big/ \left[\frac{1-p^{\Omega+1}}{1-p}-1\right] \tag{11}$$

$$=[1-p^{\omega-1}] \Big/ [1-p^{\Omega}], \tag{12}$$

which can be simplified, under the condition that $\Omega \to \infty$, to become

$$P(W<\omega)=1-p^{\omega-1}. \tag{13}$$

Hence, the probability of exceedence for the Strahler order is

$$P(W\geqslant\omega)=p^{\omega-1}, \tag{14}$$

which is a *geometric distribution* with parameter $(1-p)$. For a large total network order, the magnitude as a function of order obeys the asymptotic relation

$$m(\omega) \sim R_B^{\omega}.$$

Inverting this relationship the order can be expressed in terms of the magnitude as

$$\omega(m) \equiv \frac{\ln(m)}{\ln(R_B)}. \tag{15}$$

Inserting Eq. (15) into Eqs. (14) and (7) gives the relation

$$P(M\geqslant m) \propto p^{\{\ln(m)/\ln(R_B)\}-1}. \tag{16}$$

Recalling the definition of p, Eq. (16) can be simplified to

$$\begin{aligned} P(M\geqslant m) &= \left(\frac{1}{R_B}\right)^{\ln(m)/\ln(R_B)-1} \\ &= R_B R_B^{-\ln(m)/\ln(R_B)} \\ &= R_B R_B^{-\ln_{R_B}(m)} \\ &= R_B(1/m), \end{aligned} \tag{17}$$

indicating that the scaling exponent should have the value $\phi = 1.0$.

III. TAIL PROBABILITIES FOR COMPLETE STRAHLER BASIN AREAS FOR SELF-SIMILAR NETWORKS WITH VARIABLE TOPOLOGIES

By exploiting the equivalence of probability statements in deterministic self-similar networks, we have shown that the tail probability for complete Strahler basin areas has an exponent of $\phi = 1.0$. However, natural networks exhibit topological variability, in which sub-basins of the same Strahler order have different branching structures. We derive below the convergence to a power law of the tail probabilities for complete Strahler basin areas for a large class of underlying basin area distributions. This derivation is valid under the assumptions that the distributions of the complete Strahler basin area exhibit distributional simple scaling, and the application of Horton's law of stream numbers. The DSS of complete Strahler basin areas has been observed in natural networks [22]. In addition, the RSN model predicts DSS for complete basin areas. We express distributional simple scaling as

$$\frac{A_\omega^{(\Omega)}}{R_A^{\omega-1}} \stackrel{d}{=} A_1 \sim F_A, \tag{18}$$

where $A_\omega^{(\Omega)}$ is the statistical distribution of complete basin area at scale ω, R_A is the Horton area ratio, A_1 is the underlying distribution of basin area, which we denote as F_A, and $\stackrel{d}{=}$ denotes the equality in distribution. We will also use the empirically observed relationship between the number of streams at different scales, referred to as Horton's law of stream numbers, [see Eq. (1)]. Given the distributional simple scaling for complete Strahler basin areas, we pick a complete stream at random, with probability $N_\omega^{(\Omega)}/N_T^{(\Omega)}$, where

$$N_\omega^{(\Omega)} = N_1^{(\Omega)}(R_b^{-1})^{\omega-1},$$

$$N_T^{(\Omega)} = \sum_{\omega=1}^{\Omega} N_\omega^{(\Omega)}. \tag{19}$$

Here $N_T^{(\Omega)}$ is the total number of streams in the network. The resulting distribution of all complete basin areas $A_T^{(\Omega)}$ is a mixture over the order area distributions

$$P[A_T^{(\Omega)}>u]=\sum_{\omega=1}^{\Omega} P[A_T^{(\Omega)}>u|A_T^{(\Omega)}=A_\omega^{(\Omega)}]P[A_T^{(\Omega)}=A_\omega^{(\Omega)}] \tag{20}$$

$$=\sum_{\omega=1}^{\Omega} \frac{N_\omega^{(\Omega)}}{N_T^{(\Omega)}}P[A_\omega^{(\Omega)}>u]. \tag{21}$$

Substitution from Eqs. (18) and (19) gives

$$P[A_T^{(\Omega)}>u]=\frac{N_1^{(\Omega)}}{N_T^{(\Omega)}}\sum_{\omega=1}^{\Omega}(R_b^{-1})^{\omega-1}\left[1-F_A\left(\frac{u}{R_A^{\omega-1}}\right)\right] \tag{22}$$

$$=\frac{N_1^{(\Omega)}}{N_T^{(\Omega)}}\sum_{\omega=0}^{\Omega-1}(R_b^{-1})^{\omega}[1-F_A((R_A^{-1})^{\omega}u)]. \tag{23}$$

We rewrite Eq. (22) in the limit $\Omega\to\infty$,

$$P[A_T^{(\Omega)}>u]\equiv\frac{N_1^{(\Omega)}}{N_T^{(\Omega)}}S(u)=\frac{N_1^{(\Omega)}}{N_T^{(\Omega)}}\sum_{i=0}^{\infty}r^{-i(1+\gamma)}G(ur^{-i}), \tag{24}$$

where

$$G(x)=1-F_A(x), \tag{25}$$

$$r=R_A, \tag{26}$$

$$r^{1+\gamma}=R_b, \tag{27}$$

$$1+\gamma=\frac{\ln R_b}{\ln R_A}, \tag{28}$$

and we identify $\omega=i$ for notational convenience. We demonstrate in the Appendix that for a class of underlying distribution functions G, except for a term involving small harmonic corrections, the sum $S(u)$ in Eq. (24), and hence the tail probability for complete Strahler basin areas, takes the asymptotic form

$$S(u)\sim\frac{1}{u^{1+\gamma}}, \tag{29}$$

predicting that the tail probabilities for complete basin areas exhibit a power-law behavior with the exponent

$$\phi=\frac{\ln R_b}{\ln R_A}. \tag{30}$$

Figure 1 shows the production of a power-law tail behavior through DSS and Horton's law of stream numbers. Individual distributions of basin area $A_\omega^{(\Omega)}$ at each scale ω are shown, colored according to scale. The individual distributions of basin area are rescaled versions of the underlying distribution $F_A=A_1$ which is gamma distributed and they do not exhibit a power-law tail behavior. Each individual distribution is sampled $N_\omega^{(\Omega)}$ times and the samples are collected into a single distribution, as shown in the figure. The composite distribution shows an excellent power-law tail behavior over a large range of scales. In the figure, $R_A=R_b=4.2$ and $N_\omega^{(\Omega)}$ is chosen to be the integer closest to $R_b^{\Omega-\omega}$. The value of the scaling power-law scaling exponent is close to 1.0, as indicated by the equality of R_b and R_A. Similar simulations with $R_b\neq R_A$ show a very good correspondence with the expression given in Eq. (30).

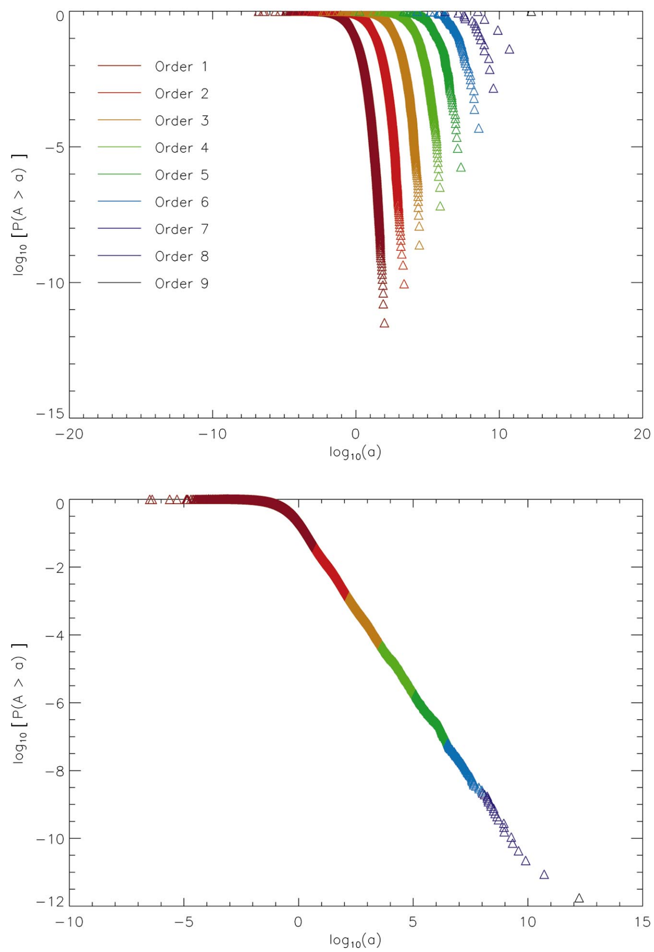

FIG. 1. (Color) Demonstration of the generation of power-law tails in the probability of exceedences. Each rescaled distribution is shown (here gamma distributed), colored according to its scale (Strahler order). The number of points in the individual distributions represent the number of samples according to Horton's law of stream numbers. The samples are gathered together in a single distribution which is seen to exhibit a power-law behavior.

Figure 2 shows the power-law tail behavior for complete Strahler basins in the Flint river in Georgia, USA. The Flint basin covers $\approx 6500\ \mathrm{km}^2$ and has a total order 7 when extracted from the USGS 1° digital elevation models. The power-law behavior is observed over a large range of scales. Figure 2 also shows the tail behavior for upstream areas above all links in the network for comparison. Note that there is also a range of scales over which an approximate power-law behavior is seen. However, there is more deviation from the power-law behavior and a pronounced dropoff at large scales compared to complete Strahler basins.

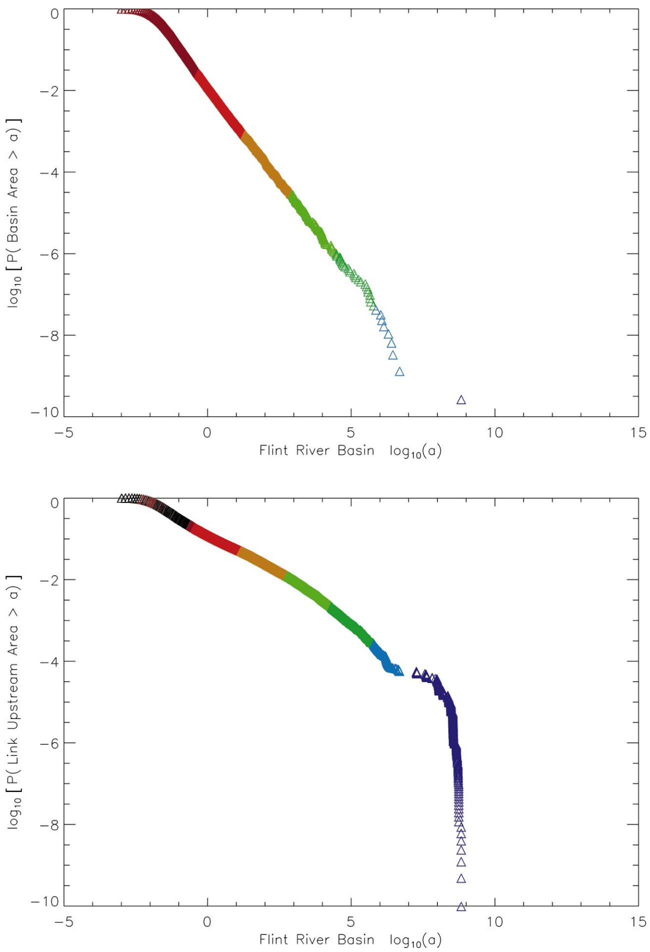

FIG. 2. (Color) Power-law tail behavior for the probability of exceedences for complete Strahler basins in the Flint river basin. The slope is consistent with that predicted by the analytic theory. For comparison, the probability of exceedences for all links in the Flint river basin is shown in the lower figure.

Table I shows a comparison of various predicted and observed values for four large basins in the United States. In the table, empirically calculated Horton ratios for a number of streams R_b, basin area R_A, basin magnitude R_M, and number of links per stream R_C are provided for comparison. ϕ_A is the estimated scaling exponent for the tail probabilities for basin area, both calculated according to Eq. (30), while ϕ_M is the scaling exponent for basin magnitude, calculated in a manner similar to Eq. (30), but with magnitude taking the place of area. The exponent ϕ is the empirically estimated basin area scaling exponent for complete Strahler basins. As can be seen from the table, the basin area Horton ratio R_A is generally greater than R_b and R_M, by as much as about 10%. However, the predictions for the scaling exponents ϕ_A and ϕ_M, based on the analysis of this section, are close to the observed values.

TABLE I. Comparison of tail probability variables for four empirical basins.

Basin	R_b	R_A	R_M	R_C	ϕ_A	ϕ_M	ϕ
1	4.45	4.90	4.62	2.61	0.94	0.98	0.91
2	4.54	4.70	4.43	2.59	0.98	1.02	0.94
3	4.40	4.85	4.52	2.41	0.94	0.98	0.93
4	4.56	4.67	4.42	2.45	0.98	1.02	0.99

IV. SUMMARY

Understanding the key geometric properties of river networks is essential for understanding how the physical processes work to shape land surfaces. One recently observed property is the power-law scaling behavior of the tail probabilities for cumulative upstream areas, which is an indicator of the scale invariance of land surface topography. We examined here the underlying statistical and scaling structure of complete Strahler basin areas, which gives rise to a power-law scaling in tail probabilities. We found that when only complete Strahler basins are considered, power-law tail probabilities for the upstream area arise as a consequence of distributional simple scaling of basin areas coupled with Horton's law of stream numbers. The scaling exponent for complete basins is well predicted by the ratio of the logarithms of the bifurcation ratio to the Horton ratio for basin areas.

The existence of power-law scaling as a direct consequence of distributional simple scaling may indicate that the power-law scaling observed for areas above every link in a network can also be understood in a simple scaling framework. Such a formulation remains an open problem.

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APPENDIX

We show here that the sum given in Eq. (24) asymptotically takes the form

$$\sum_{i=0}^{\infty} r^{-i(1+\gamma)}G(ur^{-i}) \sim \frac{1}{\ln r}\frac{1}{u^{1+\gamma}}\Bigg[M[G;1+\gamma] + 2\sum_{n=1}^{\infty} u^{-2ni\pi/\ln r} M\left(G;1+\gamma+\frac{2ni\pi}{\ln r}\right)\Bigg], \tag{A1}$$

where $G(ur^{-i})=1-F_A(ur^{-i})$ is the underlying cumulative distribution function (CDF) and $M[G;z]$ is the Mellin transform:

$$M[G;z]=\int_0^{\infty} x^{z-1}G(x)dx. \tag{A2}$$

The derivation follows the similar lines of Ref. [31], but is a generalization from the case where $G(x)=e^{-x}$ which they derived. Under certain assumptions on the functional form of G explained below, the function $M[G;1+\gamma]=O(1)$, while $M[G;1+\gamma+2ni\pi/\ln r]=O(e^{-n})$ as $n\to\infty$. Thus, the sum in question converges to a power law with exponent $1+\gamma$. We begin by following the argument of Ref. [31], using a Sommerfeld-Watson transformation, to show that

$$S(u)\sim I(u)=\frac{1}{2\pi i}\oint_C \frac{\pi\cos\pi z}{\sin\pi z}f(z)dz, \tag{A3}$$

where

$$f(z)=r^{-z(1+\gamma)}G(ur^{-z}). \tag{A4}$$

Changing variables to $\rho=r^{-z}=\xi e^{i\omega}$ Eq. (A3) can be written as

$$I(u)=\frac{-1}{2i\ln r}\int_0^{e^{i\omega}}\pi\cot(\pi\ln\rho/\ln r)\rho^{\gamma}G(u\rho)d\rho, \tag{A5}$$

which is a contour integral along the path for which $\rho=\xi e^{i\omega}$ as ξ varies from 0 to 1, and ω is a fixed angle obeying $0<\omega<\pi/2$. We shall indicate below how the angle ω is to be selected. We define

$$\delta(\rho)=\rho^{2i\pi/\ln r}=\xi^{2i\pi/\ln r}e^{-2\omega\pi/\ln r}. \tag{A6}$$

The cot term in Eq. (A5) can be written as

$$-i\frac{1+\delta(\rho)}{1-\delta(\rho)}. \tag{A7}$$

Noting that $|\delta(\rho)|=e^{-2\pi\omega/\ln,r}<1$, we can expand the cot term as

$$\cot(\pi\ln\rho/\ln r)=-i\left[1+2\sum_{n=1}^{\infty}\rho^{2ni\pi/\ln r}\right]. \tag{A8}$$

Thus,

$$I(u)=\frac{-1(-i\pi)}{2\pi\ln r}\int_0^{e^{i\omega}}\left[1+2\sum_{n=1}^{\infty}\rho^{2ni\pi/\ln r}\right]\rho^{\gamma}G(u\rho)d\rho \tag{A9}$$

$$=\frac{1}{2\ln r}\left[\int_0^{e^{i\omega}}\rho^{\gamma}G(u\rho)d\rho+2\sum_{n=1}^{\infty}I_n(u)\right], \tag{A10}$$

where

$$I_n(u)=\int_0^{e^{i\omega}}\rho^{2ni\pi/\ln r}\rho^{\gamma}G(u\rho)d\rho. \tag{A11}$$

Considering $w=u\rho$,

$$I_n(u)=u^{-(1+\gamma+2ni\pi/\ln r)}\int_0^{ue^{i\omega}}w^{\gamma+2ni\pi/\ln r}G(w)dw. \tag{A12}$$

Using the definition of the Mellin transformation given in Eq. (A2) we have

$$I_n(u)=u^{-(1+\gamma+2ni\pi/\ln r)}\{M[G;1+\gamma+2ni\pi/\ln r]-\zeta_n(u)\}, \tag{A13}$$

where

$$\zeta_n(u)=\int_{ue^{i\omega}}^{\infty}w^{\gamma+2ni\pi/\ln r}G(w)dw \tag{A14}$$

which vanishes as $u\to\infty$. Here, we have used the fact that, under the restrictions we impose on G below, the Mellin transform of G can be evaluated by rotating the path of integration from the positive real line onto the ray $\zeta e^{i\omega}$, $0\leqslant\zeta<\infty$. Finally, we obtain

$$I(u)\sim\frac{1}{2\ln r}\frac{1}{u^{1+\gamma}}\left\{M[G;1+\gamma]+2\sum_{n=1}^{\infty}u^{-2ni\pi/\ln r}\times M\left[G;1+\gamma+\frac{2ni\pi}{\ln r}\right]\right\}. \tag{A15}$$

We now show that the term $M[G;1+\gamma+2ni\pi/\ln r]$ decays exponentially in the limit of large n. It is known, Theorem 4.7.2 [35] that for any $\epsilon>0$, $x>-\alpha$,

$$M[G;x+iy]=O(\exp[-(\eta-\epsilon)|y|]),\quad |y|\to\infty \tag{A16}$$

under the following assumptions on G (here, $x=1+\gamma$ and $y=2n\pi/\ln r$): (1) $G(t)$ is analytic; (2) $G(t)=O(t^{\alpha})$, $t\to 0+$; (3) $G(t)\sim e^{-dt^{\nu}}\Sigma_{m=0}^{\infty}\Sigma_{n=0}^{N(m)}C_{mn}(\ln t)^n t^{-\gamma_m}, t\to\infty$; where $d>0, \nu>0, \gamma_m\uparrow\infty$, and $N(m)$ finite for each m. In Eq. (A16) we define η by

$$\eta=\min\left(\eta_0,\frac{\pi}{2\nu}\right), \tag{A17}$$

where the assumptions on G are valid in the open sector $s(\eta_0)$, defined by

$$s(\eta_o)=\{t|t\neq 0,|\arg(t)|<\eta_0\}. \tag{A18}$$

We have made a further assumption that d,α, and γ_m are real since we are concerned here with probability distributions. For the class of functions that we are interested in here, where $G=1-F_A$ is a complimentary CDF, the second assumption on G is satisfied with $\alpha=0$ because $G(0)=1=t^0$. The angle ω above may be set to $\eta-\epsilon$ for $\epsilon>0$. The validity of the path rotation in evaluating the Mellin transform is demonstrated in the proof of Theorem 4.7.2 in Ref. [35].

Examples

(1) Let the underlying distribution be exponentially distributed, $G(x)=e^{-x}$. It is well known that the Mellin transform of an exponential function is a Γ function; $M[e^{-x};z]=\Gamma(z)$. As $t\to 0, G(t)=O(1)$. Thus, $\alpha=0$. As $t\to\infty, G(t)\sim e^{-t}$ so $d=\nu=1$. Hence, $\eta=\pi/2$ and $M[G;1+\gamma$

$+2ni\pi/\ln r] = O[\exp(-(\pi/2-\epsilon)2n\pi/\ln r)], \forall\ \epsilon \geqslant 0$. The authors of Ref. [31] show that for the case where $G(x)=e^{-x}$, such that $|\Gamma(1+\gamma+2ni\pi/\ln r)| = O(\exp(-\pi^2 n/\ln r))$, i.e., the case here with $\epsilon=0$.

(2) Let the underlying distributions be gamma distributed with parameters (λ,k). The cumulative distribution function is then given by

$$G(x)=\int_x^\infty \frac{\lambda^k w^{k-1}}{\Gamma(k)} e^{-\lambda w} dw. \tag{A19}$$

Substituting $t=\lambda w$,

$$G(x)=\int_{\lambda x}^\infty \frac{e^{-t}t^{k-1}}{\Gamma(k)} dt \tag{A20}$$

$$=\frac{1}{\Gamma(k)}\Gamma(k,\lambda x), \tag{A21}$$

where

$$\Gamma(k,x)=\int_x^\infty e^{-t}t^{k-1}dt \tag{A22}$$

is the incomplete Γ function with parameter k. The incomplete Γ function has the asymptotic expansion

$$\Gamma(k,x)=x^{k-1}e^{-x}\sum_{m=0}^\infty (-1)^m \frac{\Gamma(1-k+m)}{\Gamma(1-k)} \frac{1}{x^m}, \tag{A23}$$

so we have

$$G(x)\sim \frac{\lambda^{k-1}}{\Gamma(k)} x^{k-1}e^{-\lambda x}\sum_{m=0}^\infty \left(\frac{-1}{\lambda}\right)^m \frac{\Gamma(1-k+m)}{\Gamma(1-k)} \frac{1}{x^m}. \tag{A24}$$

This expansion holds in the sector $s(\eta_0)$ for $\eta_0=3\pi/2$. As $x\to\infty$,

$$G(x)\sim e^{-\lambda x}\sum_{m=0}^\infty C_m x^{k-1-m}, \tag{A25}$$

so comparing Eq. (A25) with the assumptions on G, we find that $d=\lambda, \nu=1, N(m)=0$, and $\gamma_m=(m+1-k)$. Therefore, $\eta=\pi/2$, and $M[G;1+\gamma+2ni\pi/\ln r]$ decays exponentially as in the first example. We now explicitly calculate the Mellin transform for the underlying Γ distributed area distributions. $G(x)$ is given in Eq. (A19) as

$$G(x)=\int_x^\infty \frac{\lambda^k w^{k-1}}{\Gamma(k)} e^{-\lambda w} dw. \tag{A26}$$

It is known [36] that given a function $f(x)$ with Mellin transformation $M[f(x);z]$, that

$$M[g(x);z]=z^{-1}M[f(x);z+1], \tag{A27}$$

where

$$g(x)=\int_x^\infty f(t)dt. \tag{A28}$$

So setting $f(x)=x^{k-1}e^{-\lambda x}$,

$$M[f(x);z]=\int_o^\infty f(x)x^{z-1}dx \tag{A29}$$

$$=\int_0^\infty x^{z+k-2}e^{-\lambda x}dx \tag{A30}$$

$$=\lambda^{-(z+k-1)}\int_0^\infty w^{z+k-2}e^{-w}dw \tag{A31}$$

$$=\lambda^{-(z+k-1)}\Gamma(z+k-1), \tag{A32}$$

where we have made use of the substitution $w=\lambda x$. Thus, by Eq. (A27)

$$M[g(x);z]=z^{-1}\lambda^{-(z+k)}\Gamma(z+k) \tag{A33}$$

and thus we obtain

$$M[G;z]=z^{-1}\lambda^{-z}\frac{\Gamma(z+k)}{\Gamma(k)}. \tag{A34}$$

The aim here is to show that the tail probability asymptotically exhibits power-law behavior in the variable u. Referring to Eq. (A1), we note that the sum contains harmonic terms in u, and we now illustrate that these terms are for a special case small, indicating that deviation from power-law behavior is small. Noting the expression for $M[G;z]$, given in Eq. (A34), we have

$$\frac{M[G;1+\gamma+2ni\pi/\ln r]}{M[G;1+\gamma]} = \frac{(1+\gamma)\lambda^{-2ni\pi/\ln r}}{(1+\gamma+2ni\pi/\ln r)} \frac{\Gamma(k+1+\gamma+2ni\pi/\ln r)}{\Gamma(k+1+\gamma)}, \tag{A35}$$

which can be written, using the Weierstrass form for the Γ function [37] as

$$\frac{\Gamma(x+iy)}{\Gamma(x)}=\frac{xe^{-iCy}}{x+iy}\prod_{\ell=1}^\infty \frac{e^{iy/\ell}}{1+\dfrac{iy}{x+\ell}}, \tag{A36}$$

where $C=0.577\,216\ldots$ is the Euler-Mascheroni constant. The magnitude of the ratio of terms is thus

$$\left|\frac{\Gamma(x+iy)}{\Gamma(x)}\right| = \frac{|x|}{\sqrt{x^2+y^2}}\prod_{\ell=1}^\infty \frac{1}{\sqrt{1+\left(\dfrac{y}{x+\ell}\right)^2}} \tag{A37}$$

$$= \prod_{\ell=0}^{\infty} \frac{1}{\sqrt{1+\left(\frac{y}{x+\ell}\right)^2}}, \tag{A38}$$

and so

$$\left|\frac{M[G;1+\gamma+2ni\pi/\ln r]}{M[G;1+\gamma]}\right| = \frac{\lambda}{\sqrt{1+\left(\frac{2n\pi/\ln r}{1+\gamma}\right)^2}} \prod_{\ell=0}^{\infty} \frac{1}{\sqrt{1+\left(\frac{2n\pi/\ln r}{k+1+\gamma+\ell}\right)^2}}. \tag{A39}$$

The deviation from power-law behavior depends on the magnitude of

$$2\sum_{n=1}^{\infty} u^{-2ni\pi/\ln r} \frac{M[G;1+\gamma+2ni\pi/\ln r]}{M[G;1+\gamma]}, \tag{A40}$$

and the magnitude of individual terms in this sum can be evaluated using Eq. (A39). Because all the terms in the infinite product in Eq. (A39) are less than unity, evaluation of this product with any finite number of terms yields an upper bound. Using typical values,

$$2\pi/\ln r \approx 1.3, \tag{A41}$$

$$1+\gamma = \phi_A \approx 0.95, \tag{A42}$$

and taking $\lambda=1$ and $k=2$, an upper bound retaining five terms in the product is $|M[G;1+\gamma+2ni\pi/\ln r]/M[G;1+\gamma]| < 0.48$ $(n=1)$, <0.16 $(n=2)$, and <0.06 $(n=3)$. For larger n, the values approach zero rapidly using Eq. (A16), indicating that the harmonic terms cause only small deviation from power-law behavior.

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